Recent Advances in Loop Quantum Cosmology

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Outline

- Purpose of Loop Quantum Gravity
- Introduction to Loop Quantum Cosmology
- Alternative Dynamics of LQC
- Effective Scenarios of LQC
- Other Advances in LQC
1. Purpose of Loop Quantum Gravity

- **Motivations of Quantum Gravity**

  From the beginning of last century to now, two fundamental theories of physics, QM and GR, have destroyed the coherent pictures of the physical world.

  - **Classical Gravity - Quantum Matter Inconsistency**
    \[ R_{\alpha\beta}[g] - \frac{1}{2} R[g] g_{\alpha\beta} = \kappa T_{\alpha\beta}[g]. \]  
    \( R_{\alpha\beta}[g] - \frac{1}{2} R[g] g_{\alpha\beta} = \kappa T_{\alpha\beta}[g]. \)  
    \( (1) \)

  - **Singularity in General Relativity**

  - **Infinity in Quantum Field Theory**

  - **The interpretation of black hole thermodynamics**

    \[ S_{BH} = A r_{BH} / 4 \hbar. \]  
    \( S_{BH} = A r_{BH} / 4 \hbar. \)  
    \( (2) \)
The Basic Ideas of LQG

★ Combine the basic principles of GR and QM.

★ The choice of the algebra of field functions to be quantized:
   Not the positive and negative components of the field modes as in conventional QFT;
   but the holonomies of the gravitational connection and the electric flux.

★ The physical motivation of using holonomies:
   Background independence.
   The relevant variables do not refer to what happens at a point, but rather refer to the
   relation between different points connected by a line,

\[ A(c) = \mathcal{P} \exp \left( - \int_0^1 [A^i_a \dot{c}^a \tau_i] \, dt \right). \] (3)

Classical Connection Dynamics of GR

– GR can be cast into a connection dynamical formalism on a spatial 3-manifold \( \Sigma \)
  [Ashtekar 1986, Barbero 1995], where the configuration is a \( su(2) \) connection \( A^i_a \) and
  conjugate momentum is a densitized triad \( E^a_i \).
– The Hamiltonian density \( \mathcal{H}_{tot} \) is a linear combination of first-class constraints.
• The Kinematical Framework of LQG

  – The kinematical Hilbert space: \( \mathcal{H}_{\text{kin}} = L^2(\overline{A}, d\mu^0) \) is well defined on the space \( \overline{A} \) of generalized connections.
  – Uniqueness Theorem [LOST, 2005].
  – Geometric operators with discrete spectrum:
    Area operator [Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997]; Volume operator [Ashtekar and Lewandowski, 1995, 1997; Rovelli and Smolin, 1995]; Length operator [Thiemann 1998; YM, Soo, Yang, 2010]; \( \hat{Q} \) operator [YM and Ling, 2000].

• The Quantum Dynamical Issues

  – Both Gaussian constraint and spatial diffeomorphism constraint are successfully implemented at quantum level [ALMMT, 1995].
  – Hamiltonian constraint operators can be well defined in \( \mathcal{H}_{\text{kin}} \) or \( \mathcal{H}^G \) [Thiemann 1997]. Master constraint operators can be well defined in \( \mathcal{H}_{\text{Diff}} \) [Thiemann, 2003, 2005; Dittrich and Thiemann, 2004; Han and YM, 2005].
2. Introduction to Loop Quantum Cosmology

- The idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried on in the symmetry-reduced models, known as Loop Quantum Cosmology.
- One freezes all but a finite number of degrees of freedom by imposing symmetries. The simplified framework provides a simple arena to test ideas and constructions.
- Symmetries: homogeneity and (or) isotropy.
- Example: Spatially flat FRW universe
  - Spatial 3-manifold: $\mathbb{R}^3$
  - Isometry: Euclidean group
The Kinematical Setting of LQC

– One has to introduce an elementary cell $\mathcal{V}$ and restricts all integrations to this cell.
– Fix a fiducial flat metric $q_{ab}$ and denote by $V_o$ the volume of $\mathcal{V}$ in this geometry.

The gravitational phase space variables — the connections and the density weighted triads — can be expressed as

\[ A^i_a = c V_o^{-(1/3)} q^i_a \quad \text{and} \quad E^a_i = p V_o^{-(2/3)} \sqrt{q} e^a_i, \]

where $(q^i_a, e^a_i)$ are a set of orthonormal co-triads and triads compatible with $q_{ab}$ and adapted to $\mathcal{V}$.
– $p$ is related to the scale factor $a$ via $|p| = V_o^{2/3} a^2$.
– The fundamental Poisson bracket is given by: $\{ c, p \} = \kappa \gamma / 3$,
  where $\kappa = 8\pi G$.
– The gravitational part of the Hamiltonian constraint reads

\[ C_{\text{grav}} = -6c^2 \sqrt{|p|} / \gamma^2. \]
• To pass to the quantum theory, one constructs a kinematical Hilbert space

\[ \mathcal{H}_{\text{kin}}^{\text{grav}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \],

where \( \mathbb{R}_{\text{Bohr}} \) is the Bohr compactification of the real line and \( d\mu_{\text{Bohr}} \) is the Haar measure on it.

• There exists no operator corresponding to \( c \), while holonomy operators are well defined.

• The Improved Scheme:

  – It is convenient to introduce new conjugate variables by a canonical transformation:

    \[ b := \frac{\sqrt{\Delta}}{2} \frac{c}{\sqrt{|p|}}; \quad \nu := \frac{4}{3\sqrt{\Delta}} sgn(p) |p|^\frac{3}{2}, \]

    where \( \Delta \ (\sim 4\sqrt{3}\pi\gamma l_p^2) \) is the smallest non-zero eigenvalue of area operator in full LQG.

  – In the kinematical Hilbert space \( \mathcal{H}_{\text{kin}}^{\text{grav}} \), eigenstates of \( \hat{\nu} \), which are labeled by real numbers \( v \), constitute an orthonormal basis as: \( \langle v_1 | v_2 \rangle = \delta_{v_1,v_2} \).

  – The fundamental operators act on \( |v\rangle \) as: \( \hat{\nu} |v\rangle = (8\pi\gamma l_p^2/3) v |v\rangle \) and \( e^{ib} |v\rangle = |v + 1\rangle \).
3. Alternative Dynamics for LQC

- **APS Dynamics**

  The gravitational part of the APS Hamiltonian operator was given in the $v$ representation by [Ashtekar, Pawlowski, Singh, 2006]:

  $$\hat{C}_{\text{grav}} |v\rangle = f_+(v)|v + 4\rangle + f_o(v)|v\rangle + f_-(v)|v - 4\rangle.$$  \hspace{1cm} (4)

  To identify a dynamical matter field as an internal clock, one takes a massless scalar field $\phi$ with Hamiltonian $C_\phi = |p|^{-\frac{3}{2}} p_\phi^2/2$, where $p_\phi$ denotes the momentum of $\phi$.

- **Alternative Dynamics**

  LQC Gravitational Hamiltonian operator with Lorentz and Euclidean terms [Yang, Ding, YM, 2009]:

  $$\hat{H}_{\text{grav}}^F |v\rangle = F'_+(v)|v + 8\rangle + f'_+(v)|v + 4\rangle + (F'_o(v) + f'_o(v)) |v\rangle$$

  $$+ f'_-(v)|v - 4\rangle + F'_-(v)|v - 8\rangle.$$  \hspace{1cm} (5)

  The new proposed Hamiltonian constraint operator $\hat{H}_{\text{grav}}^F$ contains more terms with step of different size comparing to the original APS Hamiltonian operator.
4. Effective Scenarios of LQC

• Effective Hamiltonian and Friedmann Equation

– We can further obtain an effective Hamiltonian of $\hat{H}_F = \hat{H}_F^{\text{grav}} + \hat{H}_\phi$ with the relevant quantum corrections of order $\epsilon^2, 1/v^2 \epsilon^2, \hbar^2/\sigma^2 p_\phi^2$ as

\[
\mathcal{H}_\text{eff}^F = -\frac{3^2 \sqrt{6}}{2^3} \frac{\hbar^{1/2}}{\gamma^{3/2} \kappa^{1/2}} L |v| \left( \sin^2(2b) \left( 1 - (1 + \gamma^2) \sin^2(2b) \right) + 2\epsilon^2 \right) \\
+ \left( \frac{\kappa \gamma \hbar}{6} \right)^{3/2} \frac{|v|}{L} \rho \left( 1 + \frac{1}{2|v|^2 \epsilon^2} + \frac{\hbar^2}{2\sigma^2 p_\phi^2} \right),
\]

(6)

where $\rho = \frac{1}{2} \left( \frac{6}{\kappa \gamma \hbar} \right)^3 \left( \frac{L}{|v|} \right)^2 p_\phi^2$ is the density of the matter field.

– The modified Friedmann equation can then be derived as:

\[
H_F^2 = \left( \frac{\dot{v}}{3v} \right)^2 = \frac{\kappa}{3} \frac{\rho_c}{4(1 + \gamma^2)^2} \left( 1 - \sqrt{1 - \chi_F} \right) \left( 1 + 2\gamma^2 + \sqrt{1 - \chi_F} \right) (1 - \chi_F),
\]

(7)

where

\[
\chi_F = 4(1 + \gamma^2) \left( \frac{\rho}{\rho_c} \left( 1 + \frac{1}{2|v|^2 \epsilon^2} + \frac{\hbar^2}{2\sigma^2 p_\phi^2} \right) - 2\epsilon^2 \right).
\]

(8)
Effective Scenarios

- In the leading order effective description, when energy density of the scalar field reaches to the critical energy density $\rho^F_c = \rho_c / 4(1 + \gamma^2)$, the universe bounces from the contracting branch to the expanding branch.

- Moreover, the minus sign in front of the $\epsilon^2$ term in the expression (8) of $\chi_F$ may lead to a qualitatively different scenario from the leading order effective theory.

- The concrete form of quantum fluctuations or the Gaussian spread $\epsilon$ plays a key role here.

- A simple setting could be $\epsilon = \lambda(r)v^{-r(\phi)}$, where $0 \leq r(\phi) \leq 1$ and the parameter $\lambda(r)$ has to be suitably chosen for different value of $r$.

- For $r = 0$, besides the quantum bounce when the matter density $\rho$ increases to the Planck scale, the universe would also undergo a recollapse when $\rho$ decreases to $\rho^F_{\text{coll}} \approx 2\epsilon^2\rho_c$.

It is easy to see from Eqs.(8) and (7) that an expanding universe would undergo the recollapse and become cyclic provided $0 \leq r < 1$ asymptotically.
FIG. 1: The effective dynamics represented by the observable $v|_{\phi}$ are compared to classical trajectories. In this simulation, the parameters were: $G = \hbar = 1$, $p_{\phi} = 10000$, $\epsilon = 0.001$, $\sigma = 0.01$ with initial data $v_o = 100000$. 
FIG. 2: The cyclic model is compared with expanding and contracting classical trajectories. In this simulation, the parameters were: $G = \hbar = 1$, $p_\phi = 10000$, $\epsilon = 0.001$, $\sigma = 0.01$ with initial data $v_o = 100 000$. 
5. **Other Advances in LQC**

- **Inflation in LQC**
  
  - The loop quantum cosmological modification of Friedmann equation implies a phase of super-inflation immediately after the bounce [Bojowald 2002].
  
  - For a wide class of potentials, the super-inflation funnels the phase space trajectories to initial conditions which virtually guarantee a slow roll inflation with 60 or more e-foldings [Ashtekar, Sloan, 2009, 2011].

- **Path-integral formulation of LQC**
  
  - Concrete evidence in support of the general paradigm underlying spin foam models is provided by LQC [Ashtekar, Campiglia, Henderson, 2009, 2010].
  
  - The vertex expansion of LQC path-integral is a convergent series [Henderson, Rovelli, Vidotto, Wilson-Ewing, 2010].
  
  - The first-order effective Hamiltonians of LQC can be derived from the path-integral formalism [Ashtekar, Campiglia, Henderson, 2010; Huang, Ma, Qin, 2011]

- **Phenomenological issues**
References

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Thank you!